Variational methods for inverse scattering problems

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Recent Trends in Applied Inverse Problems 2010

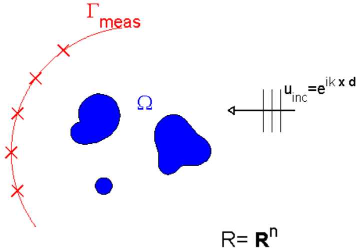
Outline

Description of the problem

Medium $\mathcal{R}$ with obstacles Ω: How many? how big? where? physical properties in Ω?

Some applications

- Medicine (tumors, fracture)
- Geophysics (oil, gas)
- Materials (damage, cracks)

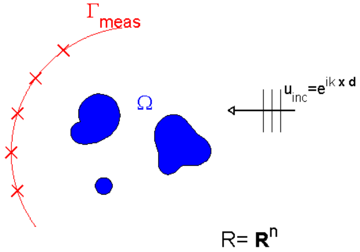


Scattering problem

An incident acoustic radiation $u_{inc} = e^{ik\mathbf{x}\cdot\mathbf{d}}$ interacts with a medium $\mathcal{R}$ containing objects Ω.

Forward (direct) problem

- The shape, size, location and physical properties of the objects are known
- Compute the response of the system at the detectors "×"
- A well–posed problem: it has a unique solution that depends continuously on the data



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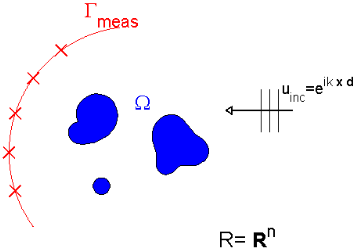

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Inverse problem

- Measurements u_{meas} are taken at the receptors
- Find the scatterers Ω and the interior parameters s.t.

$$u = u_{meas} \quad \text{on } \Gamma_{meas}, \quad \text{u= sol. forward problem}$$

- An ill–posed problem: it may not have a solution and if it has one, it may not depend continuously on the data

A simple forward problem

Ω is a penetrable known obstacle. The incident field generates a scattered wave u_{sc} in $\mathbb{R}^n \setminus \Omega$ and a transmitted wave u_{tr} in Ω. The total field

$$u = u_{inc} + u_{sc} \text{ in } \mathbb{R}^n \setminus \Omega \quad \text{and} \quad u = u_{tr} \text{ in } \Omega$$

solves

$$\begin{cases} \Delta u + k_e^2 u = 0 & \text{in } \mathbb{R}^n \setminus \Omega \\ \Delta u + k_i^2 u = 0 & \text{in } \Omega \\ u^- = u^+, \quad \partial_n u^- = \partial_n u^+ & \text{on } \partial\Omega \\ \lim_{r\to\infty} r^{(n-1)/2}(\partial_r(u - u_{inc}) - ik_e(u - u_{inc})) = 0 \end{cases}$$

where $k_e, k_i > 0$ are known

- Other boundary conditions can be handled in a similar way:

$$\begin{cases} \Delta u + k_e^2 u = 0 \quad \text{in } \mathbb{R}^n \setminus \Omega \\ \partial_n u^+ = 0 \quad \text{on } \partial\Omega \\ \lim_{r\to\infty} r^{(n-1)/2}(\partial_r(u - u_{inc}) - ik_e(u - u_{inc})) = 0 \end{cases}$$

- Non-constant parameters

$$\begin{cases} \nabla \cdot (\alpha_e(\mathbf{x})\nabla u) + k_e^2(\mathbf{x})u = 0 \quad \text{in } \mathbb{R}^n \setminus \Omega \\ \nabla \cdot (\alpha_i(\mathbf{x})\nabla u) + k_i^2(\mathbf{x})u = 0 \quad \text{in } \Omega \\ u^- = u^+, \quad \alpha_e(\mathbf{x})\partial_n u^- = \alpha_i(\mathbf{x})\partial_n u^+ \quad \text{on } \partial\Omega \\ \lim_{r\to\infty} r^{(n-1)/2}(\partial_r(u - u_{inc}) - i\kappa_e(u - u_{inc})) = 0 \end{cases}$$

where $k_s(\mathbf{x}) \geq k_{s,0} > 0$, $\alpha_s(\mathbf{x}) \geq a_{s,0} > 0$, $s = e, i$ and $k_e(\mathbf{x})/\sqrt{\alpha_e(\mathbf{x})} = \kappa_e$, $|\mathbf{x}| > R$

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Outline

1. Inverse scattering problem
2. Topological derivative methods
   - TD for shape reconstruction
   - Iterative methods
   - TD for shapes and parameters
3. Conclusions

Constrained optimization

Original problem (we assume first that k_i is known)

Find Ω such that

$$u = u_{meas} \quad \text{on } \Gamma_{meas}$$

A weaker formulation

Find Ω minimizing

$$J(\Omega) = \frac{1}{2} \int_{\Gamma_{meas}} |u - u_{meas}|^2$$

for u solving the forward problem in $\mathbb{R}^n \setminus \Omega$, Ω

- The domain Ω is the variable
- The Helmholtz transmission problem is the constraint

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Some alternatives

Modified gradient methods: differ on how an initial guess is deformed from one iteration to the next in such a way that the cost functional decreases

- Classical deformations following a vector field
  - Problem: The number of scatterers has to be known from the beginning
  - Kirsch 1993, Hettlich 1995, Potthast 1996
- Level set based deformations allow changes in topology
  - Problem: Slow evolution. Initial guess?
  - Santosa 1996, Dorn 2005
- Topological derivatives
  - Provide good initial guesses
  - Fast and allow topological changes

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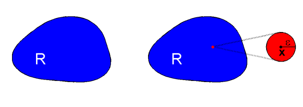
Definition of Topological Derivative (Sokowloski–Zochowski '99)

The TD of a shape functional $J(\mathcal{R})$ at a point $\mathbf{x} \in \mathcal{R}$ is

$$D_T(\mathbf{x}, \mathcal{R}) = \lim_{\varepsilon \to 0} \frac{J(\mathcal{R} \setminus B_\varepsilon(\mathbf{x})) - J(\mathcal{R})}{\text{Vol}(B_\varepsilon(\mathbf{x}))}$$



- It is a scalar function of $\mathbf{x}$
- It measures sensitivity to removing balls around a point
- $D_T(\mathbf{x}, \mathcal{R}) << 0 \implies$ high probability of finding an object

Equivalently, for $\mathbf{x} \in \mathcal{R}$ and $h(\varepsilon) = \text{Vol}(B_\varepsilon(\mathbf{x}))$

$$J(\mathcal{R} \setminus B_\varepsilon(\mathbf{x})) = J(\mathcal{R}) + h(\varepsilon) D_T(\mathbf{x}, \mathcal{R}) + o(h(\varepsilon)) \quad \text{as } \varepsilon \to 0$$

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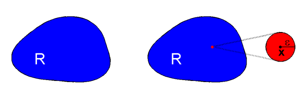

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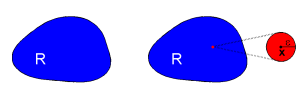

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How to obtain $D_T(\mathbf{x}, \mathcal{R})$ (Feijoo '04)

1. Given $\mathbf{x} \in \mathcal{R}$, take the ball $B_\varepsilon(\mathbf{x})$. Choose the vector field
$$\mathbf{V}(\mathbf{z}) = -\mathbf{n}(\mathbf{z}), \quad \mathbf{z} \in \partial B_\varepsilon(\mathbf{x})$$
and extend $\mathbf{V}$ to $\mathbb{R}^n$ s.t. $\mathbf{V} = \mathbf{0}$ far from $\partial B_\varepsilon(\mathbf{x})$
2. Consider the domain $\mathcal{R}_\tau := \{\mathbf{z} + \tau \mathbf{V}(\mathbf{z}) \,|\, \mathbf{z} \in \mathcal{R} \setminus B_\varepsilon(\mathbf{x})\}$. Then, $J(\mathcal{R}_\tau)$ is a scalar function of τ
3. Compute the shape derivative (Lagrangian formulation)
$$D_S := \left. \frac{d}{d\tau} J(\mathcal{R}_\tau) \right|_{\tau=0}$$
4. Use the relation (asymptotic expansions)

$$D_T(\mathbf{x}, \mathcal{R}) = \lim_{\varepsilon \to 0} \left(\frac{-1}{\mathcal{V}'(\varepsilon)} D_S \right), \quad \mathcal{V}(\varepsilon) = \mathrm{Vol}(B_\varepsilon(\mathbf{x}))$$

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Transmission problem: $u^- = u^+$, $\partial_n u^- = \partial_n u^+$

Case I: No a priori information on the obstacles, $\mathcal{R} = \mathbb{R}^n$, $\Omega = \emptyset$

Theorem. For any $\mathbf{x} \in \mathbb{R}^n$ the topological derivative of

$$J(\mathbb{R}^n) = \frac{1}{2} \int_{\Gamma_{meas}} |u - u_{meas}|^2$$

is

$$D_T(\mathbf{x}, \mathbb{R}^n) = \mathrm{Re}\left[\left(k_i^2 - k_e^2\right) u(\mathbf{x}) w(\mathbf{x})\right]$$

where u and w solve forward and adjoint problems with $\Omega = \emptyset$

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Therefore, $u = u_{inc}(\mathbf{x}) = e^{ik_e \mathbf{x}\cdot\mathbf{d}}$

Adjoint problem with $\Omega = \emptyset$:

$$\begin{cases} \Delta w + k_e^2 w = (\overline{u_{meas} - u})\delta_{\Gamma_{meas}} \quad \text{in } \mathbb{R}^n \\ \lim_{r\to\infty} r^{(n-1)/2}(\partial_r w - ik_e w) = 0 \end{cases}$$

Therefore, $w = \int_{\Gamma_{meas}} G_{k_e}(\mathbf{x}-\mathbf{y})(\overline{u_{meas} - u})(\mathbf{y})dl_{\mathbf{y}}$

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Other problems

- For Helmholtz transmission problem with $u^- = u^+$ and $\partial_n u^- = \partial_n u^+$

$$D_T(\mathbf{x}, \mathbb{R}^2) = \mathrm{Re}\left[(k_i^2 - k_e^2)u(\mathbf{x})w(\mathbf{x})\right]$$

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- For rigid obstacles

$$D_T(\mathbf{x}, \mathbb{R}^2) = \mathrm{Re}\left[2\nabla u(\mathbf{x})\nabla w(\mathbf{x}) - k_e^2 u(\mathbf{x})w(\mathbf{x})\right]$$

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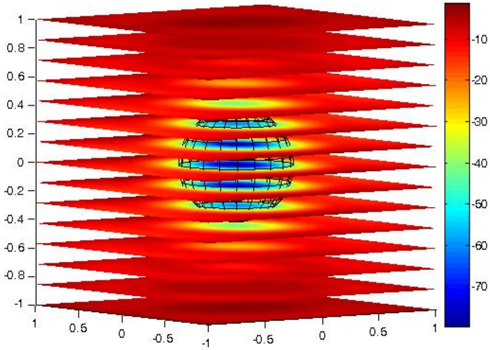
Other problems

- For the HTP with non–constant parameters

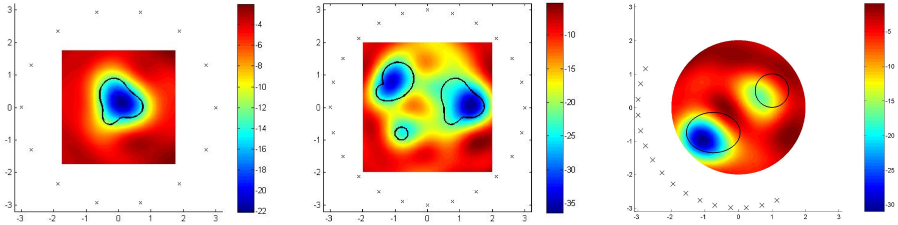
$$D_T(\mathbf{x}, \mathbb{R}^2) = \mathrm{Re}\Big[\frac{2\alpha_e(\mathbf{x})(\alpha_e(\mathbf{x}) - \alpha_i(\mathbf{x}))}{\alpha_e(\mathbf{x}) + \alpha_i(\mathbf{x})} \nabla u(\mathbf{x}) \nabla w(\mathbf{x}) + (k_i^2(\mathbf{x}) - k_e^2(\mathbf{x})) u(\mathbf{x}) w(\mathbf{x})\Big]$$

where u and w solve forward and adjoint problems. **Since k_e is non–constant, both have to be computed numerically**

Some examples



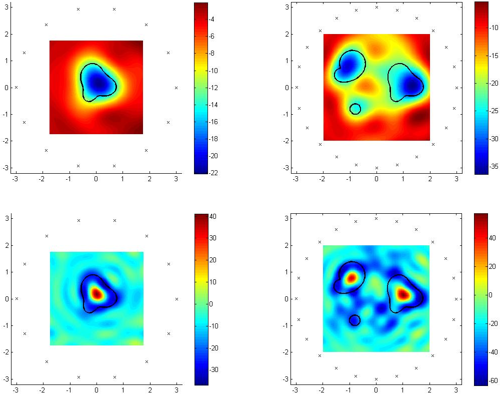
Some examples



"$\times$"= observation points, 24 incident directions in $[0, 2\pi)$,
$k_e = 2$ and $k_i = 1/2$. Level of noise=1%

Similar results when

- Observation points are further
- $+$ observation points
- $+$ incident directions
- $+$ noise

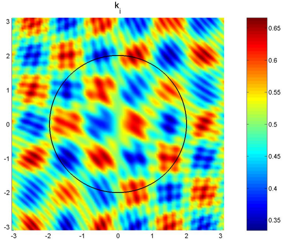


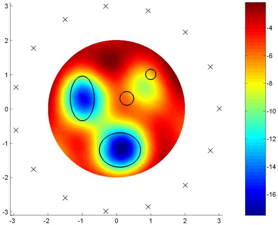
Results depend on the wave length (1 w.l.=$2\pi/k_e$):

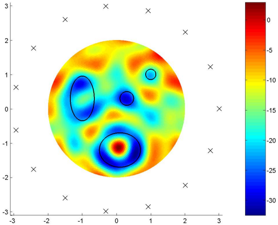
1$^{\text{st}}$ row: $k_e = 2$ and $k_i = 1/2$

2$^{\text{nd}}$ row: $k_e = 4$ and $k_i = 1$

Non–homogeneous materials







TD with an initial guess

Case II: Ω_{ap} first guess, $\mathcal{R} = \mathbb{R}^n \setminus \Omega_{ap}$, $\Omega = \Omega_{ap}$

Theorem. For any $\mathbf{x} \in \mathbb{R}^n \setminus \Omega_{ap}$ the topological derivative of

$$J(\mathbb{R}^n \setminus \Omega_{ap}) = \frac{1}{2} \int_{\Gamma_{meas}} |u - u_{meas}|^2$$

is

$$D_T(\mathbf{x}, \mathbb{R}^n \setminus \Omega_{ap}) = \mathrm{Re}\left[(k_i^2 - k_e^2)\, u(\mathbf{x}) w(\mathbf{x})\right]$$

where u and w solve forward and adjoint probl. with $\Omega = \Omega_{ap}$

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- Forward problem with $\Omega = \Omega_{ap}$:

$$\begin{cases} \Delta u + k_e^2 u = 0 \quad \text{in } \mathbb{R}^n \setminus \Omega_{ap} \\ \Delta u + k_i^2 u = 0 \quad \text{in } \Omega_{ap} \\ u^- = u^+, \quad \partial_n u^- = \partial_n u^+ \quad \text{on } \partial\Omega_{ap} \\ \lim_{r\to\infty} r^{(n-1)/2}(\partial_r(u - u_{inc}) - ik_e(u - u_{inc})) = 0 \end{cases}$$

- Adjoint problem with $\Omega = \Omega_{ap}$:

$$\begin{cases} \Delta w + k_e^2 w = (\overline{u_{meas} - u})\delta_{\Gamma_{meas}} \quad \text{in } \mathbb{R}^n \setminus \Omega_{ap} \\ \Delta w + k_i^2 w = 0 \quad \text{in } \Omega_{ap} \\ w^- = w^+, \quad \partial_n w^- = \partial_n w^+ \quad \text{on } \partial\Omega_{ap} \\ \lim_{r\to\infty} r^{(n-1)/2}(\partial_r w - ik_e w) = 0 \end{cases}$$

The boundary conditions influence u and w!

Remarks

1. If $\Omega = \Omega_{ap}$, u and w are computed numerically. For our model problem we use BEM
2. For Neumann or general transmission problems the formula when $\Omega = \Omega_{ap}$ is analogous

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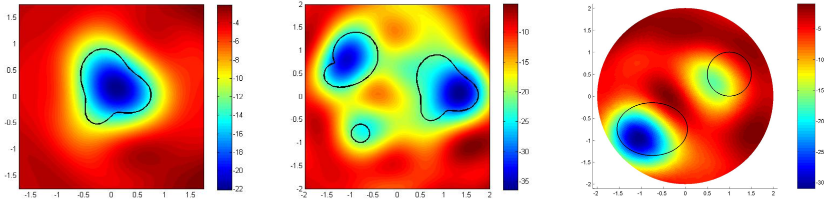
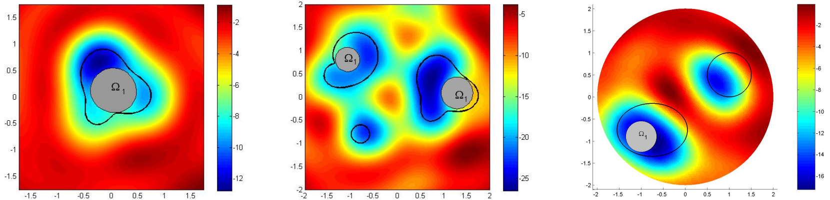


Same examples as before with $\Omega = \emptyset$



Initial guess Ω_1 superimposed on the TD when $\Omega = \Omega_1$

Outline

1. Inverse scattering problem
2. Topological derivative methods
   - TD for shape reconstruction
   - Iterative methods
   - TD for shapes and parameters
3. Conclusions

A monotone iterative method

Idea

By removing from the initial domain regions where the TD is negative and large, we obtain a new domain in which the magnitude of the topological derivative is smaller

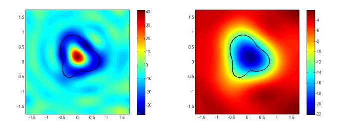
Algorithm

1. Compute the TD when $\Omega = \emptyset$
2. Take $\Omega_1 = \{\mathbf{x}, D_T(\mathbf{x}, \mathbb{R}^n) < -C_1\}, \; C_1 > 0$
3. For j=1:jmax
   - Compute the TD in $\mathbb{R}^n \setminus \Omega_j$
   - Select $\Omega_{j+1} \supset \Omega_j$

$$\Omega_{j+1} = \Omega_j \cup \{\mathbf{x}, D_T(\mathbf{x}, \mathbb{R}^n \setminus \Omega_j) < -C_{j+1}\}$$

How to choose C_j?

- First step: $\Omega_1 = \{\mathbf{x}, D_T(\mathbf{x}, \mathbb{R}^2) < -C_1\}$



$$C_1 = \frac{3}{5}|\min D_T|$$

- Accept C_1 if $J_1 < J_0$
- Otherwise $C_1' < C_1$

- Iterations: $\Omega_{j+1} = \Omega_j \cup \{\mathbf{x}, D_T(\mathbf{x}, \mathbb{R}^2 \setminus \Omega_j) < -C_{j+1}\}$

$$C_{j+1} = \frac{9}{10}|\min D_T|$$

Stopping criteria?

$$J \approx 0 \quad \text{or} \quad \Omega_j \approx \Omega_{j+1} \quad \text{or} \quad |u_\delta - u_{meas}| < 1.2\delta$$

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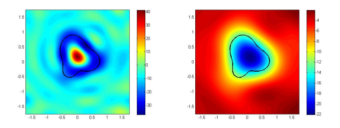

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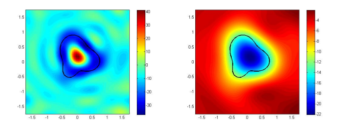

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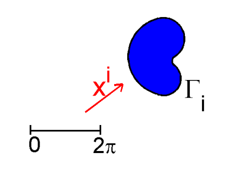
Implementation in 2D: $TD = \mathrm{Re}\left[(k_i^2 - k_e^2)\, u(\mathbf{x}) w(\mathbf{x})\right]$

- Computation of TD when $\Omega = \emptyset$:

$$u = u_{inc} \quad \text{and} \quad w = \int_{\Gamma_{meas}} G(\mathbf{x}-\mathbf{y})(\overline{u_{meas} - u})(\mathbf{y}) dl_{\mathbf{y}}$$

- Computation of TD when $\Omega = \Omega_j$:
  - u and w solve HTP with $\Omega = \Omega_j = \cup_{i=1}^{d} \Omega_j^i$
  - To apply BEM, we assume that Ω_j^i is star–shaped:



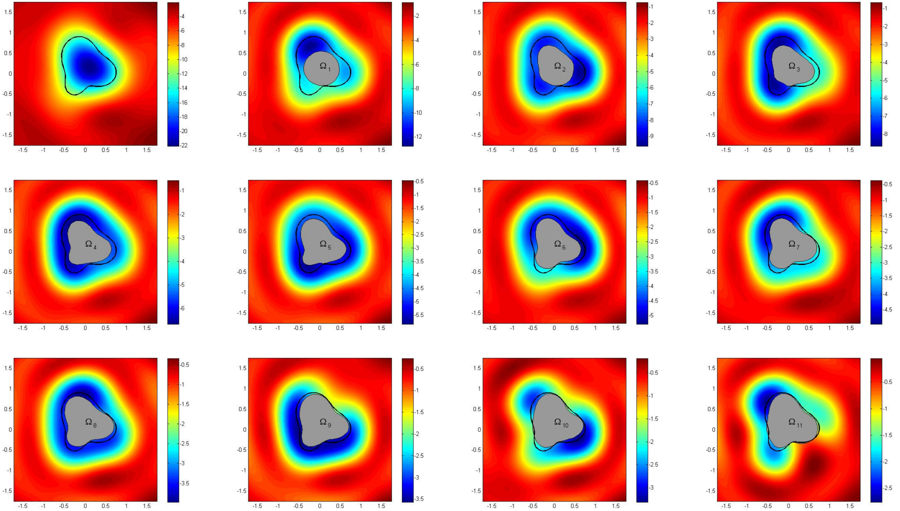
$$\mathbf{x}^i(t) = (c_x^i, c_y^i) + r^i(t)(\cos(t), \sin(t))$$

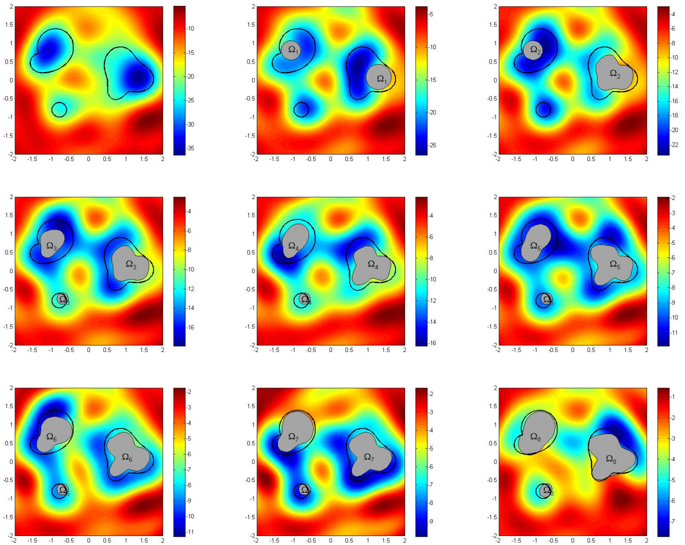
We approximate

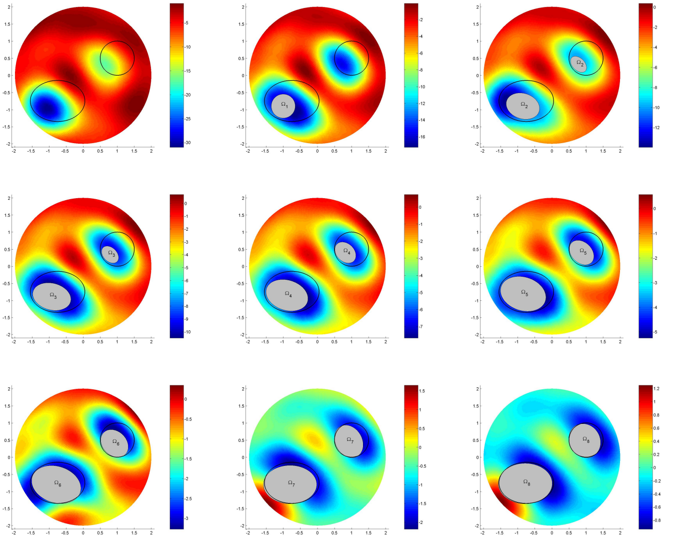
$$r^i(t) \approx a_0^i + \sum_{k=1}^{K} (a_k^i \cos(kt) + b_k^i \sin(kt))$$

  - Solve a least squares problem to obtain $a's, b's$ such that

$$\Omega_j = \Omega_{j-1} \cup \{\mathbf{x}, TD_{j-1} << 0\}$$


Ω_1
Ω_2
Ω_3
Ω_4
Ω_5
Ω_6
Ω_7
Ω_8
Ω_9
Ω_{10}
Ω_{11}


Ω1
Ω2
Ω3
Ω4
Ω5
Ω6
Ω7
Ω8


Ω_1
Ω_2
Ω_2
Ω_3
Ω_3
Ω_4
Ω_4
Ω_5
Ω_5
Ω_6
Ω_6
Ω_7
Ω_7
Ω_8
Ω_8

A non–monotone iterative method

- The previous scheme was monotone: it generates a sequence of increasing approximations. If at some step a spurious region is included, it cannot be removed
- Non–monotone schemes need a generalized definition of topological derivative:
    - $\mathbf{x} \in \mathbb{R}^n \setminus \Omega_{ap}$,

$$D_T(\mathbf{x}, \mathbb{R}^n \setminus \Omega_{ap}) = \lim_{\varepsilon \to 0} \frac{J(\mathbb{R}^n \setminus \Omega_{ap}) - J(\mathbb{R}^n \setminus (\Omega_{ap} \cup B_\varepsilon(\mathbf{x})))}{Vol(B_\varepsilon(\mathbf{x}))}$$

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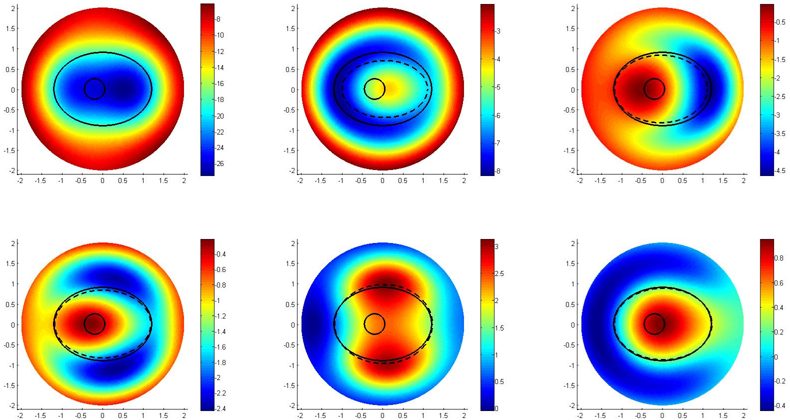
Algorithm

1. Compute the TD when $\Omega = \emptyset$ (as before)
2. Take $\Omega_1 = \{\mathbf{x}, D_T(\mathbf{x}, \mathbb{R}^n) < -C_1\}, \; C_1 > 0$ (as before)
3. For j=1:jmax
   - Compute the TD in $\mathbb{R}^n$ when $\Omega = \Omega_j$
   - Select Ω_{j+1}

$$\text{if } \mathbf{x} \in \mathbb{R}^n \setminus \Omega_j \text{ and } TD < -C_j \implies \mathbf{x} \in \Omega_{j+1}$$

$$\text{if } \mathbf{x} \in \Omega_j \text{ and } TD > C_j' \implies \mathbf{x} \notin \Omega_{j+1}$$



Reconstruction of an annular region. Points are added or removed at each step. The hole is recovered after 6 steps

Outline

Direct and inverse problems

Direct problem

$$\begin{cases} \Delta u + k_e^2 u = 0 \quad \text{in } \mathbb{R}^n \setminus \Omega \\ \Delta u + k_i^2 u = 0 \quad \text{in } \Omega \\ u^- = u^+, \quad \partial_n u^- = \partial_n u^+ \quad \text{on } \partial\Omega \\ \lim_{r\to\infty} r^{(n-1)/2}(\partial_r(u - u_{inc}) - ik_e(u - u_{inc})) = 0 \end{cases}$$

Inverse problem

Find Ω and k_i

Idea

- In the first computation of the TD, i.e. when $\Omega = \emptyset$, we do not need to know k_i:

$$D_T(\mathbf{x}, \mathbb{R}^2) = \mathrm{Re}\left[(k_i^2 - k_e^2) u(\mathbf{x}) w(\mathbf{x})\right]$$

  where $u = u_{inc}$ and $w = \int_{\Gamma_{meas}} G_{k_e}(\mathbf{x} - \mathbf{y})(\overline{u_{meas} - u})(\mathbf{y}) dl_{\mathbf{y}}$
- We compute the TD taking $k_i^0 \approx k_e$ to get an initial guess Ω_1
- In the next step, we update k_i by a gradient method
- Update Ω, update k_i

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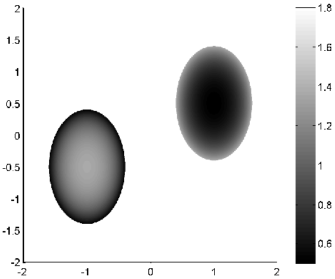
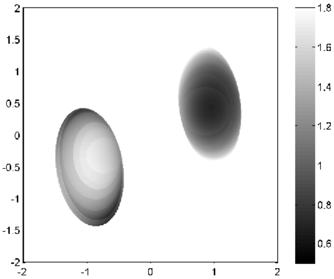
Materiales heterogéneos

Original



Reconstruction



Outline

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Conclusions

- We have computed topological derivatives for inverse scattering problems involving Helmholtz equations with constant and non–constant parameters
- The TD gives a good approximation of the number, size and location of objects buried in the medium
- An iterative procedure improves their shape, and catches smaller objects, if missed in the first trial
- An extended notion of topological derivative allows to design non–monotone schemes, useful to capture holes
- The algorithm for shape reconstruction can be combined with a gradient method to recover both shapes and parameters

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More information

- A Carpio, ML Rapún. *Solving inhomogeneous inverse problems by topological derivative methods*. Inverse Problems 24 (2008) Art. 045014
- A Carpio, ML Rapún. *An iterative method for parameter identification and shape reconstruction*. Inv Probl Sci Eng 18 (2010) 35–50

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María–Luisa Rapún: `marialuisa.rapun@upm.es`

Thank you!